METRIC SPACES: RE-EXAM 2018

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Problem 1 (5+5%). Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a non-empty metric space, r and s be two positive radii, and $\mathsf{B}_r^{\mathrm{d}_{\mathcal{X}}}(x) = \mathsf{B}_s^{\mathrm{d}_{\mathcal{X}}}(y)$ for some $x, y \in \mathcal{X}$.

- Is it true that $r = s$?
- Is it true that $x = y$?

Problem 2 (15%). Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a metric space. For all $x, y \in \mathcal{X}$ put $\varrho(x, y) = \ln(1 + \mathrm{d}_{\mathcal{X}}(x, y))$ by definition. Prove that the function $\varrho \colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is another metric on $\mathcal{X}$.

Problem 3 (15%). Is it always true that the closure $\overline{\mathsf{B}_r^{\mathrm{d}_{\mathcal{X}}}(a)}$ of an open disk of radius r coincides with the set $\{x \in \mathcal{X} \mid \mathrm{d}_{\mathcal{X}}(x, a) \leqslant r\}$? (state and prove, e.g., by counterexample)

Problem 4 (20%). Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a metric space and $\{U_i \mid i \in \mathcal{I}\}$ be a family of connected subsets $U_i \subseteq \mathcal{X}$ such that $U_i \cap U_j \neq \varnothing$ for all $i, j \in \mathcal{I}$. Prove that the union $U = \bigcup_{i \in \mathcal{I}} U_i$ is connected.

Problem 5 (25%). Let A and B be compact subsets of a Hausdorff space $\mathcal{X}$. Prove that the intersection $A \cap B$ is compact in $\mathcal{X}$.

Problem 6 (15%). Solve for $x(s)$ the integral equation,

$$x(s) = \tfrac{1}{2} \int_0^1 x(t)\,\mathrm{d}t + \exp(s) - \frac{e}{2} + \frac{1}{2},$$

by consecutive approximations starting from $x_0(s) = 0$. (In the end, verify by a direct substitution that the solution which you have found satisfies the equation.)

Date: 27 June 2018. GOOD LUCK !